Helicity invariants in 3D : kinematical aspects

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Abstract

Exact, degenerate two-forms $\Theta_k = d\theta_k$ on time-extended space $R \times M$ which are invariant under the unsteady, incompressible fluid motion on three-dimensional region M are introduced. The equivalence class up to exact one-forms of θ_k is splitted by the velocity field. The components of this splitting corresponds to Lagrangian and Eulerian conservation laws for helicity densities. These are expressed as the closure of three-forms $\theta_k \wedge \Theta_l$ which depend on two discrete and a continuous parameter. Each Θ_k is extended to a symplectic form on $R \times M$. The subclasses of θ_k's giving rise to Eulerian helicity conservations is shown to result in conformally symplectic structures on $R \times M$. The connection between Lagrangian and Eulerian conservation laws for helicity is shown to be the same as the conformal equivalence of a Poisson bracket algebra to infinitely many local Lie algebra of functions on $R \times M$.

1 Introduction

In this work, we shall concern with the problem of constructing infinitely many helicity type integrals for three dimensional incompressible fluids analogous to the enstrophy type Casimirs of two dimensional flows. We shall express Lagrangian and Eulerian conservation laws using invariant differential forms constructed for each kinematical (particle relabelling) symmetry of the velocity field. The relation between these two types of conservation laws will be shown to be equivalent to the conformal relation between Poisson and Jacobi structures on time-extended space of flows. The invariants under consideration are related to the description of reduced phase space of Eulerian equations of ideal fluids which are the orbits of coadjoint action of the group $Diff_{vol}(M)$ of volume preserving diffeomorphisms of the flow domain M.

1.1 The problem of coadjoint orbits

The motion of an ideal fluid on a Riemannian manifold M can be formulated as geodesic motion with respect to a right-invariant metric on $Diff_{vol}(M)$ [1],[2]. The Lie algebra $\mathcal{X}_{div}(M)$ consists of divergence-free vector fields on M tangential to the boundary of M. The dual $\mathcal{X}^*_{div}(M)$ of the Lie algebra is the space $\Lambda^1(M)/dC^\infty(M)$ of non-exact one-forms on M which can be identified with $\mathcal{X}_{div}(M)$ via L^2-inner product [3]. In particular, identifying the velocity field v with the one-form $v^\flat$ via the isomorphism defined by the metric of M, the dynamical formulation of an ideal fluid as geodesic motion on $Diff_{vol}(M)$ can be reduced to the Lie-Poisson dynamics

$$\frac{\partial v^\flat}{\partial t} = -ad^*_{\delta H/\delta v^\flat}(v^\flat) \tag{1}$$

on $\mathcal{X}_{div}(M)$ [1]-[7]. The physical problem of describing the reduced phase space of the Eulerian equations (1) of hydrodynamics, or the equivalent mathematical problem of the description of orbits of coadjoint action for the group of volume preserving diffeomorphisms involves intersections of infinitely many Casimir functionals of the Lie-Poisson structure which, by definition, satisfy

$$\{H, C\}_{+LP}(v) = -\int_M v \cdot [\frac{\delta H}{\delta v}, \frac{\delta C}{\delta v}]\, \mu_M = 0 \tag{2}$$

for all functionals H on the reduced phase space. That means, they are invariants of any dynamics described by Eq.(1) on coadjoint orbits and hence

characterize the reduced phase space rather than the reduced dynamical equations [4],[8],[9].

For (+)Lie-Poisson structure described by the bracket in Eq.(2), the Casimirs are left invariant functions on orbits associated with the right action of $Diff_{vol}(M)$. In fluid mechanical context, the right action corresponds to the particle relabelling symmetries while the left action generates the motion [6],[7]. This group theoretical description of motion is essentially independent of the dimension of the flow space M. In spite of this fact, qualitatively different results were obtained for Casimirs of even and odd dimensional flows [8]-[14].

Proposition 1 *Let M be a Riemannian manifold of dimension m with a volume μ_M. The Eulerian equations (1) have infinitely many generalized enstrophy type integrals*

$$I_\Phi(v) = \int_M \Phi((d_M v^\flat)^n/\mu_M)\mu_M \tag{3}$$

if $m = 2n$ and, for $m = 2n+1$ there exist at least one generalized helicity invariant

$$I(v) = \int_M v^\flat \wedge (d_M v^\flat)^n \tag{4}$$

where d_M is the exterior derivative on M and $v^\flat$ is the one-form obtained by lowering the indices of v by the metric on M.

Apart from incompressible fluids, the physical framework of this result has been shown in Ref. [8], with slight modifications, to include the equations of superconductivity [15],[12], barotropic fluids [16] and ideal magnetohydrodynamics [17],[18].

It has been concluded in Ref. [8] that there might be the possibility of connecting the integral invariants (3) and (4) to symplectic properties of the space of trajectories of the velocity field. In fact, for two-dimensional flows, this connection is well-understood in the framework of the natural symplectic structure of M defined by its volume two-form. In this case, one can represent the generators of the symmetry algebra $\mathcal{X}_{div}(M)$ by Hamiltonian vector fields and hence it can be identified with the space of nonconstant functions on M endowed with the canonical Poisson bracket (see section (3.1 and Refs. [3],[19]). It then follows that for each infinitesimal symmetry of the velocity

field one can associate a continuous family of Casimirs of the form of (3) depending on its Hamiltonian (or stream) function [20].

In this work, relying on a symplectic set-up analogous to the one in two-dimensions, we shall construct infinite families of helicity integrals for three dimensional flows. The geometric framework to be employed will also enable us to investigate the kinematical properties of invariants in the context of Jacobi structures which includes Poisson, symplectic, conformally symplectic and contact structures as particular cases.

1.2 Content of the work

In Ref. [21], we introduced, for incompressible flows on a three-dimensional region M of Euclidean space, a symplectic structure on $R \times M$. Using the automorphism algebra of this structure we obtained, in Ref. [22], the generators of volume preserving diffeomorphisms on M and showed that they can, as in two dimensions, be represented by Hamiltonian vector fields. This enabled us to express the Lie-Poisson bracket through the Poisson bracket of invariant functions on M. In this work, we shall utilize these results, which will be summarized in the next section, to construct infinite families of helicity invariants and to obtain a kinematical interpretation of them in the framework of particle relabelling symmetries.

In section (3), associated to each infinitesimal symmetry we shall introduce invariant two-forms Θ_k which are closed and degenerate. This will enable us to express conservation laws globally as the closure of the three-forms $\theta_k \wedge \Theta_l$ where θ_k are potential one-forms satisfying $d\theta_k = \Theta_k$. The densities which are conserved at each point of trajectories will be called Lagrangian. By an Eulerian conservation law we shall mean a divergence expression in which the integral over fluid domain of a density is conserved [23]. We shall show that the type of conservation laws is determined by different classes of one-forms θ_k characterized by their orientation and invariance properties with respect to the flow of the velocity field.

In section (4), we shall give a characterization of the connection between Lagrangian and Eulerian conservations of helicity in the framework of Jacobi structures on $R \times M$. We shall first extent the two-forms Θ_k to symplectic forms on $R \times M$ without altering their invariance properties. We shall then

establish the correspondences

$$\begin{array}{c} relative \\ invariants \end{array} \leftrightarrow \begin{array}{c} Eulerian \\ conservation \\ laws \end{array} \leftrightarrow \begin{array}{c} conformally \\ symplectic \\ structures \end{array} \tag{5}$$

characterized by invariance with respect to flow. Each family is parametrized by functions on $R \times M$ not in the kernel of $\partial_t + v$. Moreover, we shall show that they are conformally equivalent to the corresponding families in the relations

$$\begin{array}{c} absolute \\ invariants \end{array} \leftrightarrow \begin{array}{c} Lagrangian \\ conservation \\ laws \end{array} \leftrightarrow \begin{array}{c} symplectic \\ structures \end{array} . \tag{6}$$

We shall conclude that the kinematical interpretation of invariants of coadjoint orbits are connected with the conformal properties of the space of trajectories, and these can be understood better in the framework of local Lie algebraic structures on the function spaces over $R \times M$ rather than with the geometry of $Diff_{vol}(M)$.

2 Kinematical symmetries

The motion of an incompressible fluid in Lagrangian coordinates can be described as geodesic motion on the group $Diff_{vol}(M)$ of volume preserving diffeomorphisms of M via left action by evaluation. The right action of the group generates the particle relabelling symmetries. A divergence-free frozen in vector field can be used to cast the suspended velocity field on $R \times M$ into Hamiltonian form. Under certain conditions automorphisms of the symplectic structure can be identified with the infinitesimal time-dependent symmetries on $R \times M$ of the velocity field. The velocity field itself separates the infinitesimal symmetries into generators of reparametrizations and diffeomorphisms of M. All these generators can be realized as Hamiltonian vector fields. These results will be summarized from Refs. [21] and [22].

2.1 Kinematical description and symplectic structure

Let the open set $M_0 \subset R^3$ be the domain occupied initially by an incompressible fluid and $x(t = 0) = x_0 \in M_0$ be the initial position, i.e., a

Lagrangian label. For a fixed initial position x_0, the Eulerian coordinates $x(t) = g_t(x_0)$ define a smooth curve in R^3 describing the evolution of fluid particles. For each time $t \in I \subset R$, the volume preserving embedding $g_t : M_0 \to g_t(M_0) = M \subset R^3$ describes a configuration of fluid. A flow is then a curve $t \mapsto g_t$ in the group $Diff_{vol}(M)$ of volume preserving diffeomorphisms. The time-dependent Eulerian (spatial) velocity field v_t that generates g_t is defined by

$$\frac{dx}{dt} = \frac{dg_t(x_0)}{dt} = (v_t \circ g_t)(x_0) = v(t,x) \tag{7}$$

where $v_t \circ g_t$ is the corresponding Lagrangian (material) velocity field [1],[6],[7]. Since g_t is volume preserving, $v_t(x)$ is a divergence-free vector field over M and Eq.(7) is a non-autonomous dynamical system associated with it. The Lagrangian description of fluid motion is the description by trajectories [6],[24]-[27], that is, by solutions of non-autonomous ordinary differential equations (7) or, equivalently, by solutions of the autonomous system represented by the suspended velocity field

$$\partial_t + v(t,x) \ , \qquad v = \mathbf{v} \cdot \nabla \tag{8}$$

on the time-extended space $I \times M$.

The velocity field is right invariant. Hence, the generators of the right action which form the infinite dimensional left Lie algebra of $Diff_{vol}(M)$ are infinitesimal particle relabelling symmetries. The dynamical formulation on $T^*Diff_{vol}(M)$ when reduced by these symmetries results in the (+)Lie-Poisson structure on $\mathcal{X}^*_{div}(M)$. The Eulerian dynamics on the coadjoint orbits is determined by a right-invariant functional on $I \times M$. The Eulerian dynamical equations can be used to construct a formal symplectic structure for (8) on a time-extended domain $I \times M$ [21],[22].

Proposition 2 *In the Eulerian description of motion of an incompressible fluid in 3D let the dynamics of the velocity field* $\mathbf{v}$ *be governed by*

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{F} \tag{9}$$

and assume that the divergence-free vector field B *and the function* φ *satisfy*

$$\frac{\partial B}{\partial t} + [v, B] = 0 \ , \quad \frac{\partial \varphi}{\partial t} + v(\varphi) = 0 \tag{10}$$

which are the frozen-field equations. Then, $\partial_t + v$ is a Hamiltonian vector field with the symplectic two-form

$$\Omega = \omega + \sigma \wedge dt \ , \quad \sigma = i(v)(\omega) - d_M\varphi \ , \quad \omega = \mathbf{B} \cdot (d\mathbf{x} \wedge d\mathbf{x}) \tag{11}$$

and the Hamiltonian function φ. Here, $i(v)(\cdot)$ is the interior product with the vector field v. The invariant density in the symplectic volume

$$\mu \equiv \frac{1}{2}\Omega \wedge \Omega = \rho_\varphi dt \wedge dx \wedge dy \wedge dz \tag{12}$$

is given by $\rho_\varphi \equiv B(\varphi)$

Following Ref. [22] we shall show that the symplectic set-up of proposition (2) for three dimensional flows is an appropriate modifications of natural geometric tools of two dimensional flows in the sense that it enables us to construct the generators of volume preserving diffeomorphisms and to represent them by Hamiltonian vector fields on M.

2.2 Reparametrization and particle relabelling symmetries

A time-dependent vector field $U = \xi\partial_t + u$ on $I \times M$ is an infinitesimal geometric symmetry of the Lagrangian motion on M described by v if the criterion

$$[\partial_t + v, \xi\partial_t + u] = (\xi_{,t} + v(\xi))(\partial_t + v) \tag{13}$$

is satisfied. These are the most general symmetries of the system (7) of first order ordinary differential equations [29]. Starting with the Hamiltonian vector fields

$$u_0 \equiv -\rho_\varphi^{-1} B \ , \quad U_1 = -u_0(h_1)(\partial_t + v) + \frac{dh_1}{dt} u_0 - \rho_\varphi^{-1} \nabla\varphi \times \nabla h_1 \cdot \nabla \tag{14}$$

associated with the symplectic two-form Ω and the functions t and h_1, where h_1 is arbitrary, one can generate infinitely many infinitesimal automorphisms of Ω. These are vector fields satisfying $\mathcal{L}_{U_k}(\Omega) = 0$ where $\mathcal{L}_{U_k}(\cdot)$ is the Lie derivative. The automorphism algebra of Ω can be identified with infinitesimal symmetries of v if $dh_1/dt = f(\varphi)$ for some function f. In this case, the vector fields

$$u_0 \ , \quad U_1 \ , \quad U_k \equiv (\mathcal{L}_{U_1})^{k-1}(u_0) \ , \quad k \geq 2 \tag{15}$$

generate an infinite hierarchy of time-dependent infinitesimal Hamiltonian symmetries of the velocity field v.

In order to relate these symmetries to the volume preserving diffeomorphisms of M it will be appropriate to adopt a coordinate independent definition of the dynamical system (7) associated with v because the velocity field is defined only implicitly by some non-linear Eulerian dynamical equations. This can be achieved by the interpretation of the system (7) as an algebraic variety $\{\dot{x} - v(t,x) = 0\} \subset J^1_{t,x,1,\dot{x}}(I \times M)$ of the first jet space over (t,x). This can be embedded into $I \times T_x M$ and thus, Eqs.(7) define a section of the first jet bundle over $I \times M$ represented by $\partial_t + v$. As for any such section, this induces the unique connection $\Gamma \equiv dt \otimes (\partial_t + v)$ on $I \times M \rightarrow I$ [22],[28]-[30]. The connection Γ on $I \times M$ dictated by the velocity field v splits the vector fields U_k of the form $\xi_k \partial_t + u_k$ into horizontal and vertical generators

$$U_k^h = \xi_k(\partial_t + v) \ , \quad U_k^v = u_k - \xi_k v \tag{16}$$

of reparametrization symmetries which are gauge transformations and of diffeomorphisms on M, respectively. Here, ξ_k's are conserved functions of the velocity field and hence we can identify the algebra of reparametrization symmetries with the kernel of $\partial_t + v$ in $C^\infty(I \times M)$. U_k^v's are divergence-free vector fields on M with respect to the time-dependent volume

$$\mu_M \equiv i(\partial_t)(\mu) = \rho_\varphi dx \wedge dy \wedge dz = -\sigma \wedge \omega = d_M \varphi \wedge \omega \tag{17}$$

on M induced from the symplectic volume if and only if h_1 is conserved under the flow of $\partial_t + v$. This greatly simplifies the form of vector fields (15) to

$$U_k = \xi_k(\partial_t + v) + W_k \ , \qquad u_k \equiv \xi_k v + W_k \tag{18}$$

where the left-invariant vector fields on M

$$W_1 \equiv \rho_\varphi^{-1} \nabla h_1 \times \nabla \varphi \cdot \nabla \ , \quad W_k \equiv (\mathcal{L}_{W_1})^{k-1}(u_0) \ , \quad k \geq 2 \tag{19}$$

are μ_M-divergence free. Introducing the time-dependent functions

$$\xi_1 \equiv -u_0(h_1) \ , \quad \xi_k \equiv -u_0(h_k) \ , \quad h_k \equiv (W_1)^{k-2}(\xi_1) \ , \quad k \geq 2 \tag{20}$$

which are in the form of potential vorticities [26] we have

$$W_k = \rho_\varphi^{-1} \nabla \varphi \times \nabla h_k \cdot \nabla \ , \quad k \geq 2 \tag{21}$$

and these satisfy the Lie bracket relations

$$[W_k, W_l] = \rho_\varphi^{-1} \nabla\varphi \times \nabla h_{lk} \cdot \nabla \equiv -W_{kl} \tag{22}$$

of the left Lie algebra of $Diff_{vol}(M)$. The invariant functions

$$h_{lk} \equiv \rho_\varphi^{-1} \nabla\varphi \cdot \nabla h_l \times \nabla h_k \tag{23}$$

and hence W_{kl} are antisymmetric in their indices. For each element of the hierarchies U_k, U_{kl} this process can be continued to find time-dependent, μ_M-divergence-free vector fields $u_0, W_k, W_{kl}, ...$ on M which commute with the suspension $\partial_t + v$.

2.3 Hamiltonian structures of symmetries

The vector fields U_k are Hamiltonian with the symplectic two-form (11) and the Hamiltonian functions h_k. For the generators W_k of volume preserving diffeomorphisms we have, from Ref. [22]

Proposition 3 *W_k's are manifestly Hamiltonian with the Nambu-Poisson type bracket*

$$\{f, g\}_\varphi = \rho_\varphi^{-1} \nabla\varphi \cdot \nabla f \times \nabla g \; , \tag{24}$$

characterized by the function φ, and with the Hamiltonian functions h_k. The closed two-forms

$$-\,\omega_k \equiv i(W_k)(\mu_M) = d_M\varphi \wedge i(W_k)(\omega) \tag{25}$$

on M can be identified with the left-invariant elements of $\mathcal{X}^*_{div}(M)$.

The first equality in Eqs.(25) is the invariant definition of the curl vector [8],[9] and it implies the Clebsch representations

$$\rho_\varphi \mathbf{W}_k = \nabla \times \varphi \nabla h_k = -\nabla \times h_k \nabla\varphi \tag{26}$$

of W_k's. Since

$$i(W_k)(\omega) = \mathbf{B} \times \mathbf{W}_k \cdot d\mathbf{x} = d_M h_k + \xi_k d_M \varphi \tag{27}$$

we also conclude from Eqs.(25) that the two-forms ω_k are exact $\omega_k = d_M \gamma_k$ for one-forms $\gamma_k \in \Lambda^1(M)/d_M C^\infty(M)$ defined up to differential of functions

on M. Conversely, since the map $d_M : \Lambda^1(M)/d_M C^\infty(M) \to Image(d_M) \subset \Lambda^2(M)$ does not depend on the representatives we have the identifications

$$[\gamma_k] \leftrightarrow \omega_k \leftrightarrow W_k \tag{28}$$

between equivalence classes of one-forms modulo exact one-forms, closed two-forms [3],[8],[7] and the generators W_k of volume preserving diffeomorphisms. The Lie bracket algebra of left-invariant vector fields W_k is isomorphic, via

$$W_{\{h_k,h_l\}} = -W_{kl} = [W_k, W_l] \ , \tag{29}$$

to the Poisson bracket algebra (24) of generalized potential vorticities on the flow space M. Analogous to the canonical Poisson bracket for two dimensional flows, the Hamiltonian structure on M of the vector fields W_k can be used to write the (+)Lie-Poisson bracket in three dimensions in terms of the Poisson bracket (24) on M.

2.4 Nilpotent generators

The Poisson bracket (24) is degenerate and possesses a Casimir function on $I \times M$. If this is one of the functions h_k for some $k > 1$, then we have, by comparing Eqs.(24) and (21), that $W_k = 0$. It follows from Eqs.(19) that

$$\mathcal{L}_{W_1}(W_l) = (\mathcal{L}_{W_1})^l(u_0) = (\mathcal{L}_{W_1})^{l-k+1}(W_k) \equiv 0 \ , \quad \forall\, l \geq k \ . \tag{30}$$

This, together with Eqs.(22) and (23) imply that W_1 is a nilpotent element of the (possibly infinite dimensional) algebra generated by the finite set $\{u_0, W_1, ..., W_{k-1}\}$ of vector fields. Then, by Jacobson-Morozov theorem [31],[32] there exist vector fields, say W_0, W_{-1}, in this finitely generated algebra satisfying the Lie bracket relations

$$[W_1, W_0] = 2W_1 \ , \quad [W_1, W_{-1}] = W_0 \ , \quad [W_{-1}, W_0] = -2W_{-1} \tag{31}$$

of the $sl(2,R)$ algebra. Even though the Casimirs of the bracket (24) gives zero functional on the orbits, the geometric structures arising from this case, that is, from the nilpotency of W_1 is non-trivial and results in Godbillon-Vey type invariants [9],[33]-[35]. We refer to Ref. [35] for an investigation of this case which requires a separate treatment, its relation with the symplectic structure Ω as well as physically relevant applications. To this end, we shall solely assume that W_1 is not a nilpotent element of $\mathcal{X}_{div}(M)$. In other words, the Casimir of (24) is different from the invariant functions h_k of potential vorticity type.

3 Helicity conservations

We shall construct invariant differential forms of the velocity field associated with the infinitesimal symmetries. We shall then express the conservation laws as closure of three-forms obtained from various combinations of invariants. The resulting divergence expressions imply that the integral over the flow domain of a density is conserved. As in Ref. [23], these will be called Eulerian conservation laws. Under certain conditions the divergence expression reduces to the vanishing of the time derivative of the density itself. That means, the density is conserved at each point of the flow domain. This will be called a Lagrangian conservation law. We shall show that the distinction between types of conservation laws is kinematical and can be characterized by gauge transformations on invariant forms.

3.1 Symplectic structure and integral invariants

We shall discuss and compare the relations between symplectic structures, Eulerian equations, infinitesimal symmetries and integral invariants for two and three dimensional flows. The ideas to be employed in the rest of this section will rely on these observations. The construction of infinite families of helicity integrals will be motivated by proposition (4) connecting the symplectic two-form Ω to the integral invariant $I(v)$.

The time-dependent, divergence-free velocity field v on a two-dimensional domain M with coordinates (x, y), and its curl vector field w perpendicular to M can be expressed by means of a function $\psi = \psi(t, x, y)$ as

$$v = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} , \quad w = \phi\frac{\partial}{\partial z} , \quad \phi \equiv \nabla^2\psi \tag{32}$$

and they satisfy the frozen-field equation for w

$$\frac{\partial\phi}{\partial t} + \{\phi, \psi\}_{can} = 0 \tag{33}$$

where $\{\ ,\ \}_{can}$ is the canonical bracket on the two-dimensional domain M. The Hamilton's equations (33) are equivalent to the Euler equations of ideal fluid in two dimensions and by the Lie algebra isomorphism (29) to the condition for w to be an infinitesimal time-dependent symmetry of v [36],[7].

We observe that the formal restriction of the symplectic two-form (11) to the vector fields (32) manifests its interplay with the two dimensional Eulerian dynamical equations. Namely, we find that the degenerate two-form

$$\Omega = -(d\varphi + \phi d\psi) \wedge dt + \phi dx \wedge dy \tag{34}$$

is closed whenever Eq.(33) holds. Moreover, the suspended velocity field $\partial_t + v$ in three dimensions is Hamiltonian provided the Hamiltonian function φ satisfies the same equation.

To reveal the connection between the symplectic structure Ω and the helicity invariant $I(v)$ of proposition (1) we shall consider, in the notation of proposition (1) or of Refs. [7]-[9], the Lie-Poisson equations (1) for the kinetic energy functional, that is, the Euler equations of ideal fluids. We recall that a differential $p-$form α is said to be a relative invariant for a vector field V if there exist a $p-1-$form β such that

$$\mathcal{L}_V(\alpha) = d\beta \; . \tag{35}$$

If $\beta = 0$, α is said to be an absolute invariant [29],[37],[38].

Proposition 4 *On a three-dimensional Riemannian manifold M the Euler equations of ideal fluids for a divergence-free vector field v tangent to the boundary of M are*

$$\frac{\partial v}{\partial t} + \nabla_v v = -grad(p) \tag{36}$$

where p is the pressure and grad is taken with respect to the metric on M. Define the exact two-form ω by

$$\omega = d_M v^\flat \equiv i(w)(\mu_M) \tag{37}$$

where the second equality is the invariant definition of the curl vector or the vorticity w. Then, ω is an absolute invariant of $\partial_t + v$, or equivalently, w is an infinitesimal symmetry of v, that is, a frozen-in field. The symplectic two-form is exact

$$\Omega = -d\theta \; , \quad \theta = (\varphi + p + \frac{1}{2}v^2)dt - v^\flat \tag{38}$$

and it is an extention to $I \times M$ of ω on M via Euler equations (36). θ and $\theta \wedge \Omega$ are relative invariants. The integrand in $I(v)$ of proposition (1) is the

scalar density in the three-form $\theta \wedge \Omega$ and is associated with the infinitesimal symmetry w of the velocity field. The identity

$$d(\theta \wedge \Omega) + \Omega \wedge \Omega \equiv 0 \; . \tag{39}$$

is an expression for the (Eulerian) conservation law of helicity in divergence form.

Proof: Using the identity $(\nabla_v v)^\flat = \mathcal{L}_v(v^\flat) - d_M v^2/2$ in Eq.(36) and taking the derivative d_M of resulting equation we obtain $\omega_{,t} + \mathcal{L}_v(\omega) = 0$ [7]. So, ω is an absolute invariant for $\partial_t + v$. In terms of $\omega = i(w)(\mu_M)$ this gives

$$\begin{aligned} 0 &= (i(w)(\mu_M))_{,t} + \mathcal{L}_v(i(w)(\mu_M)) && (40)\\ &= i(w_{,t})(\mu_M) + i(w)((\mu_M)_{,t}) + i(w)(\mathcal{L}_v(\mu_M)) + i([v,w])(\mu_M) && (41)\\ &= i(w_{,t} + [v,w])(\mu_M) + i(w)((\mu_M)_{,t} + \mathcal{L}_v(\mu_M)) && (42) \end{aligned}$$

where we used the identity $\mathcal{L}_v \circ i(w) - i(w) \circ \mathcal{L}_v = i([v,w])$ [7] in obtaining the second equation. Since μ_M is invariant, the second term in Eq.(42) vanishes and the fact that it defines a volume implies $w_{,t} + [v,w] = 0$. Thus, w is an infinitesimal symmetry of v.

Solving $v^\flat_{,t}$ from the Euler equations in the derivative of θ one obtains the symplectic two-form Ω. Equivalently, it can also be obtained from $\omega = d_M v^\flat$ by replacing d_M with d, solving the time derivative of the velocity field from the Euler equations (36) and adding the one-form φdt with φ being any conserved function of v. The Lie derivatives

$$\mathcal{L}_{\partial_t + v}(\theta) = d\chi \; , \quad \mathcal{L}_{\partial_t + v}(\theta \wedge \Omega) = d(\chi \Omega) \; , \quad \chi \equiv \varphi + p - \frac{1}{2} v^2 \tag{43}$$

express the relative invariances of θ and $\theta \wedge \Omega$. For the last conclusion, we compute

$$\begin{aligned} 0 &= d(\theta \wedge \Omega) + \Omega \wedge \Omega && (44)\\ &= d[v^\flat \wedge d_M v^\flat + [(\varphi + p + \frac{1}{2} v^2) d_M v^\flat - v^\flat \wedge (d_M \varphi - i(v)(\omega))] \wedge dt] \\ &\qquad + 2 d_M v^\flat \wedge (d_M \varphi - i(v)(\omega)) \wedge dt && (45)\\ &= [(v^\flat \wedge d_M v^\flat)_{,t} + d_M((p + \frac{1}{2} v^2) d_M v^\flat + i(v)(\omega) \wedge d_M v^\flat)] \wedge dt && (46) \end{aligned}$$

which is the divergence expression for the local form of the conservation law for the total helicity $I(v)$. •

Note that the helicity flux in Eq.(46) is independent of the function φ which we have introduced by hand to make the symplectic form non-degenerate. The function χ in the invariance expressions (43) is related, in Ref. [39], to the invariance under the particle relabelling symmetries of the Lagrangian density of the variational formulation of Eqs.(36) in which φ corresponds to the sum of the potential energies of the fluid [24],[26].

3.2 Invariant differential forms

Proposition (4) explains the connection between conservation law for the helicity integral $I(v)$ and the curl w of v viewed as an infinitesimal symmetry. It, moreover, gives a recipe to construct the conserved density in $I(v)$ starting from w. We shall now apply this to the generators W_k of particle relabelling symmetries and obtain infinitely many integrals of the form of $I(v)$. Our presentation of invariant forms will be three-fold (c.f. Eqs.(47-(49)). The abstract coordinate independent form of them (c.f. Eqs.(47)) will serve for generalization to and for computation (as in proposition (4)) on any Riemannian manifold M. The Clebsch representation of them (c.f. Eqs.(48)) will follow from our earlier results presented in section (2). The coordinate expressions in the notations of three-dimensional vector calculus (c.f. Eqs.(49)) will be used to make the results more excessible as well as to justify the invariant formulation.

Proposition 5 *The exact, degenerate two-forms*

$$\begin{aligned} \Theta_k &= \omega_k + i(v)(\omega_k) \wedge dt & (47) \\ &= d\varphi \wedge dh_k & (48) \\ &= -\rho_\varphi[\mathbf{W}_k \cdot d\mathbf{x} \wedge d\mathbf{x} + (\mathbf{W}_k \times \mathbf{v}) \cdot d\mathbf{x} \wedge dt] & (49) \end{aligned}$$

are absolute invariants of the velocity field. They are the extentions to the space $I \times M$ *of* ω_k*'s and can be obtained by replacing* d_M *in Eq.(25) by* $d = d_M + dt \wedge \partial_t$*.*

Proof: The degeneracy

$$\Theta_k \wedge \Theta_l = 0 \ , \quad \forall\, k, l \qquad (50)$$

can be seen by direct computation. The closure and absolute invariance of ω_k's imply via Eq.(47) the closure of Θ_k's. For Eq.(49) the closure of Θ_k follows from the conservations of ρ_φ, $\nabla \cdot \mathbf{v} = 0$, $\nabla \cdot (\rho_\varphi \mathbf{W}_k) = 0$ and the left invariance of W_k's. The absolute invariance follows from the closure of Θ_k and that it annihilates the extended velocity field. Employing the Poincaré lemma, we introduce potential one-forms θ_k

$$\Theta_k = d\theta_k \ , \quad -\theta_k = \psi_k dt + A_k \quad A_k \equiv \mathbf{A}_k \cdot d\mathbf{x} \tag{51}$$

where ψ_k and the representative A_k of the one-form γ_k satisfying $d_M \gamma_k = \omega_k$ are defined by the equations

$$\omega_k = d_M A_k \ , \quad A_{k,t} - i(v)(\omega_k) = d_M \psi_k \tag{52}$$

or equivalently,

$$\rho_\varphi \mathbf{W}_k = \nabla \times \mathbf{A}_k \ , \quad \frac{\partial \mathbf{A}_k}{\partial t} + \rho_\varphi \mathbf{W}_k \times \mathbf{v} = \nabla \psi_k \tag{53}$$

the first of which can be regarded as to define the Clebsch potentials [24],[26] $\mathbf{A}_k$ for the vector fields W_k. Θ_k's can be obtained from $\omega_k = d_M A_k$ by replacing d_M with d and solving A_k from Eqs.(51). •

The one-forms θ_k whose derivatives give Θ_k are defined up to differential of an arbitrary function on $I \times M$. The invariance properties of θ_k's are characterized by these functions. Since exact forms on $\mathcal{X}^*_{div}(M)$ result in zero functionals, one-forms

$$[\theta_k] \equiv \{\theta_k + d\lambda_k \mid d\theta_k = \Theta_k, \ \lambda_k \in C^\infty(I \times M)\} \tag{54}$$

constitute an equivalence class on coadjoint orbits. However, they can be distinguished by the velocity field according to their behaviour under its flow. To this end, we shall assume that the one-forms θ_k as given by Eqs.(51) and (52) are all annihilated by the extended velocity field

$$i(\partial_t + v)(\theta_k) = -\psi_k - i(v)(A_k) = 0 \quad \forall \ k \tag{55}$$

which will avoid the proliferation of various exact one-forms in the foregoing discussions. Having fixed this gauge for ω_k's, we compute

$$\mathcal{L}_{\partial_t + v}(\theta_k) = di(\partial_t + v)(\theta_k) = d\chi_k \ , \quad \chi_k \equiv \lambda_{k,t} + v(\lambda_k) \tag{56}$$

so that, if λ_k is not a conserved function for the velocity field θ_k's are only relatively invariant. Thus, the type of invariance of θ_k's separates the class (54) into subclasses

$$[\theta_k]^a \equiv \{\theta_k + d\lambda_k \mid d\theta_k = \Theta_k,\ \lambda_k \in ker(\partial_t + v) \subset C^\infty(I \times M)\} \tag{57}$$

$$[\theta_k]^r \equiv \{\theta_k + d\lambda_k \mid d\theta_k = \Theta_k,\ \lambda_k \in C^\infty(I \times M)/ker(\partial_t + v)\} \tag{58}$$

of absolutely and relatively invariant one-forms, respectively. Here, we identify elements of $ker(\partial_t+v)$ which differ by an additive term linear in the time variable. We shall take the representatives of $[\theta_k]^a$ as defined by Eqs.(51) and (52). We thus have the decomposition

$$[\theta_k] = [\theta_k]^a \oplus [\theta_k]^r\ . \tag{59}$$

of one-forms on $I \times M$ which can alternatively be interpreted as the splitting of $T^*(I \times M)$ into horizontal and vertical subspaces by the connection $dt \otimes (\partial_t + v)$.

3.3 Lagrangian conservation laws

We shall first construct helicity densities which are conserved at each point of trajectories of the velocity field. These Lagrangian conservation laws will be formulated using a pair of invariant potential one-forms one of which is in the class $[\theta_k]^a$ of absolutely invariant ones. In the case of Clebsch representations, a proper orientation of them becomes nesessary.

Proposition 6 *For $k \neq l$ and for $\theta_k \in [\theta_k]^a$, the closure of the three-forms $\theta_k \wedge \Theta_l$ is equivalent to conservations of helicity densities $\mathbf{A}_k \cdot \mathbf{W}_l$ under the flow of the velocity field.*

Proof: The three-forms are closed identically by the property (50) of the two-forms Θ_k. To obtain the conservation laws we write

$$\begin{aligned}\theta_k \wedge \Theta_l = {} & \rho_\varphi \mathbf{A}_k \cdot \mathbf{W}_l\ \ dx \wedge dy \wedge dz + \\ & \rho_\varphi[\psi_k \mathbf{W}_l + (\mathbf{v}\cdot\mathbf{A}_k)\mathbf{W}_l - (\mathbf{W}_l \cdot \mathbf{A}_k)\mathbf{v}] \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt \quad (60) \\ = {} & \rho_\varphi(\mathbf{A}_k \cdot \mathbf{W}_l)(dx \wedge dy \wedge dz - \mathbf{v} \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt) \quad (61)\end{aligned}$$

where we used Eqs.(51),(53), the vector identity $\mathbf{A}_k \times (\mathbf{W}_l \times \mathbf{v}) = (\mathbf{v}\cdot\mathbf{A}_k)\mathbf{W}_l - (\mathbf{W}_l \cdot \mathbf{A}_k)\mathbf{v}$ and Eq.(55). Applying d to (61) we get

$$\frac{\partial}{\partial t}(\mathbf{A}_k \cdot \mathbf{W}_l) + \mathbf{v} \cdot \nabla(\mathbf{A}_k \cdot \mathbf{W}_l) = 0 \tag{62}$$

which is the expression for a Lagrangian conservation law. •

Two particular solutions to Eqs.(53) are given by the one-forms

$$\theta_k^- = -h_k d\varphi \ , \quad \theta_k^+ = \varphi dh_k \tag{63}$$

which are connected with the Clebsch representations (26) of W_k's. The existence of Lagrangian conserved densities of helicity type depends on the proper choice of orientation for the potential one-form which is a topological property. For the above solutions

$$\psi_k^- = h_k \varphi_{,t} \ , \quad \psi_k^+ = -\varphi h_{k,t} \ , \quad \mathbf{A}^- = h_k \nabla\varphi \ , \quad \mathbf{A}^+ = -\varphi \nabla h_k \tag{64}$$

of Eqs.(53) which follows from Eqs.(47) the absolutely invariant three-forms

$$\theta_k^+ \wedge d\epsilon\theta_l^\epsilon = \theta_k^+ \wedge \Theta_l = \varphi dh_k \wedge d\varphi \wedge dh_l \tag{65}$$

$$\begin{aligned} &= \varphi[\nabla h_k \times \nabla\varphi \cdot \nabla h_l \;\; dx \wedge dy \wedge dz \\ &\quad \nabla h_l \times (\varphi_{,t}\nabla h_k - h_{k,t}\nabla\varphi) + h_{l,t}\nabla h_k \times \nabla\varphi) \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt] \end{aligned} \tag{66}$$

are the non-zero products of Θ_l's with θ_k's $\forall \ k \neq l$ whereas $\theta_k^- \wedge \Theta_l \equiv 0 \ \forall \ k, l$. In Eq.(66) the helicity density is recognized to be the volume density in the three space with coordinates $(h_k, h_l, \varphi^2/2)$.

3.4 Eulerian conservation laws

The conservation of helicity density at each point of trajectories is, a consequence of the absolute invariance which follows from the condition (55). The violation of the condition within the cohomology class of the potential one-forms by a gauge transformation, that is, by the addition of an exact one-form not in the kernel of $\partial_t + v$ changes the character of conservation laws. Since the operators d and $\mathcal{L}_{\partial_t+v}$ commute the absolute invariance of two-forms Θ_k are not affected by such a transformation. However, for the corresponding class of three-forms $\theta_k \wedge d\theta_l$ we have

$$\mathcal{L}_{\partial_t+v}(\theta_k \wedge d\theta_l) = d(\chi_k d\theta_l) \tag{67}$$

and hence the distinction by invariance under velocity field in the class of one-forms can be carried over to the class of three-forms. This, in turn, changes completely the character of the Lagrangian conservation laws

$$d(\theta_k \wedge d\theta_l) = 0 \tag{68}$$

of proposition (6). Namely, for each $\chi_k \neq constant$ within a given class of one-form θ_k one obtains a conservation law of divergence type.

Proposition 7 *(1) Each representative of the class $[\theta_k]$ parametrized by the function space $C^\infty(I \times M)$ gives infinitely many conservation laws of helicity type expressed as the closure of the three-forms $\theta_k \wedge \Theta_l$, $\forall\, l \neq k$.*

(2) For potential one-forms in $[\theta_k]^a$ the conservation law is of Lagrangian type (62), while for those in $[\theta_k]^r$ it is an Eulerian conservation law.

(3) For each Lagrangian invariant, there are infinitely many Eulerian conservation laws parametrized by $C^\infty(I \times M)/ker(\partial_t + v)$ all of which are equivalent within the class of defining three-forms.

(4) Each Eulerian conservation law associated with $\theta_k \wedge \Theta_l$ degenerates into the equivalent Lagrangian one whenever W_l is tangent to the level surfaces of χ_k. In this case, the functions $W_l(\lambda_k)$ are also Lagrangian conserved densities.

Proof: Representing a one-form θ_k in the class (54) by

$$\theta_k = \phi_k dt + \mathbf{a}_k \cdot d\mathbf{x} \ , \quad \phi_k = \lambda_{k,t} - \psi_k \ , \quad \mathbf{a}_k = \nabla \lambda_k - \mathbf{A}_k \tag{69}$$

we compute the three-form

$$\begin{aligned} -\theta_k \wedge d\theta_l = \rho_\varphi \mathbf{a}_k \cdot \mathbf{W}_l \;\; dx \wedge dy \wedge dz + \\ \rho_\varphi [\phi_k \mathbf{W}_l + (\mathbf{v} \cdot \mathbf{a}_k)\mathbf{W}_l - (\mathbf{a}_k \cdot \mathbf{W}_l)\mathbf{v}] \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt \end{aligned} \tag{70}$$

to be associated with the helicity type conservation laws.

(1) The closure of (70) gives

$$\frac{\partial \mathcal{H}_{kl}}{\partial t} + \nabla \cdot [\mathcal{H}_{kl} \mathbf{v} - \rho_\varphi \chi_k \mathbf{W}_l)] = 0 \tag{71}$$

for the generalized helicity densities

$$\mathcal{H}_{kl} = \rho_\varphi \mathbf{a}_k \cdot \mathbf{W}_l = \rho_\varphi (W_l(\lambda_k) - \mathbf{A}_k \cdot \mathbf{W}_l) \tag{72}$$

which depend on two discrete and a continuous parameter. For fixed k and function λ_k, $\mathcal{H}_{kl}$ are indexed by the generators W_l of particle relabelling symmetries which are infinite in number. For each fixed pair of discrete parameters k, l, the continuous parameter in $\mathcal{H}_{kl}$ is the function λ_k and this characterizes the type of conservation laws.

(2) For $\lambda_k \in ker(\partial_t + v)$, we have, from Eqs.(56) $\chi_k = 0$ and Eq.(71) turns into a Lagrangian conservation law.

(3) $\forall\ \lambda_k \in ker(\partial_t + v)$ the density $\mathcal{H}_{kl}$ in Eq.(71) is independent of this function because the term $W_l(\lambda_k)$ vanishes via commutativity of $\partial_t + v$ and W_l. As λ_k takes values in $C^\infty(I \times M)/ker(\partial_t + v)$ we obtain infinitely many Eulerian conservation laws with densities $\mathcal{H}_{kl}$ having the Lagrangian invariant $\mathbf{A}_k \cdot \mathbf{W}_l$ in common and differing only in the term $W_l(\lambda_k)$. Since we fixed the discrete parameters, these conservation laws arise from the same class of three-forms.

(4) Using divergence-free properties of v and W_l, the Eulerian conservation law (71) can be put into the form

$$\frac{\partial \mathcal{H}_{kl}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{H}_{kl} = \rho_\varphi W_l(\chi_k) \tag{73}$$

from which the last conclusion follows. If $W_l(\chi_k) = W_l((\partial_t + v)(\lambda_k)) = 0$, we have

$$\frac{\partial W_l(\lambda_k)}{\partial t} + v(W_l(\lambda_k)) = 0 \tag{74}$$

by left-invariance of W_l. $\bullet$

As we remarked earlier for the Clebsch representations of W_k's, the non-vanishing helicity densities result from the proper choice of orientation within the class of potential one-forms. This is a topological property whereas the distinction (59) in the cohomology classes of potential one-forms and hence in the type of conservation laws is purely kinematical. Moreover, since the representatives of the class $[\theta_k]$ can not be distinguished on orbits, there is no difference between Lagrangian and Eulerian conserved densities as invariants of coadjoint orbits. The gauge degree of freedom in helicity integrals has been indicated in [23]. The relation between Lagrangian and Eulerian conservation laws has not been made clear because the exact one-form in the representative of θ_k has been restricted to be an invariant of the velocity field.

Analogous to the invariants (4) of even dimensional flows, the Casimirs

$$C(\rho, w(\rho)) = \int_M \Phi(\rho, w(\rho))\ dx\ dy\ dz \tag{75}$$

depending on the arbitrary function Φ has been considered for three-dimensional motions [39]-[42]. However, contrary to (4) involving Eulerian vorticity variable, $C(\rho, w(\rho))$ contain Lagrangian information [36].

It easily follows from the symmetry condition (13) that an infinitesimal symmetry takes a conserved density into another one. The last conclusion of proposition (7), on the other hand, makes it possible to have $W_l(\lambda_k)$ as a conserved function of the velocity field even if λ_k itself is not. In fact, for negative orientation of the potential one-forms θ_k in Eqs.(63) the conserved densities $\mathcal{H}_{kl}$ consist only of these functions because we have $\mathbf{A}_k^- \cdot \mathbf{W}_l = 0$ in Eq.(72).

4 Kinematical interpretations

We shall now seek a characterization of the connection between the Eulerian conservation laws and Lagrangian invariants both of which stemm from the one and the same hierarchy of infinitesimal symmetries, with the kinematical distinction encoded in the invariance properties in the equivalence classes of the associated differential forms with respect to the flow of the velocity field. We shall find an interpretation of this distinction in the geometric language of Jacobi structures on $I \times M$ or equivalently, in the Lie algebraic structures on $C^\infty(I \times M)$. More precisely, we shall prove

Proposition 8 *The connection between Lagrangian invariants and the hierarchies of Eulerian conservation laws anchored to them is the same as the conformal equivalence of a Poisson bracket algebra to an infinite hierarchy of local Lie algebras.*

4.1 Extentions of invariant forms

To establish the result of proposition (8) we shall consider further extentions of two-forms Θ_k to closed two-forms with maximal rank on $I \times M$, that is, to symplectic forms. The degeneracy of Θ_k's can be removed by an additional term which does not change their closure and invariance properties. In particular, these local conditions are satisfied if we demand the extentions of Θ_k's to be symplectic.

Proposition 9 *Let η_k be a time-dependent, closed and left-invariant one-form on M which, for $k > 0$, is different from $d_M\varphi$ and $d_M h_k$. Then, $\partial_t + v$ is locally Hamiltonian with the symplectic two-form*

$$\Omega_k = \Theta_k + \eta_k \wedge dt \tag{76}$$

$$= \quad -[\rho_\varphi \mathbf{W}_k \cdot d\mathbf{x} \wedge d\mathbf{x} + (\rho_\varphi \mathbf{W}_k \times \mathbf{v} - \mathbf{n}_k) \cdot d\mathbf{x} \wedge dt] \ . \tag{77}$$

where we let $\eta_k \equiv \mathbf{n}_k \cdot d\mathbf{x}$. If, moreover, $\partial_t + v$ is globally Hamiltonian, then, there exists time-dependent function φ_k such that

$$\eta_k = -d_M \varphi_k \ , \quad i(\partial_t + v)(\Omega_k) = d\varphi_k \ , \quad k \geq 0 \ . \tag{78}$$

In this case, Ω_k is exact with the canonical one-form

$$\tilde{\theta}_k = \varphi_k dt - \theta_k \ , \quad \Omega_k = -d\tilde{\theta}_k \tag{79}$$

where θ_k is a representative of the class of one-forms satisfying $d\theta_k = \Theta_k$. θ_k and $\tilde{\theta}_k$ have the same invariance properties and the identities

$$d(\tilde{\theta}_k \wedge \Omega_l) + \Omega_k \wedge \Omega_l = 0 \ . \tag{80}$$

give the helicity conservation laws of proposition (7).

Proof: Using Eqs.(76),(47),(25) we compute

$$\begin{aligned} \Omega_k \wedge \Omega_l &= d_M \varphi \wedge (\eta_l \wedge d_M h_k + \eta_k \wedge d_M h_l) \wedge dt & (81) \\ &= -(\mathbf{W}_k \cdot \mathbf{n}_l + \mathbf{W}_l \cdot \mathbf{n}_k)\ \mu & (82) \end{aligned}$$

where μ is the symplectic volume defined by ω. For $k = l$ this gives

$$\Omega_k \wedge \Omega_k = -2\mathbf{W}_k \cdot \mathbf{n}_k \ \mu = -2\nabla\varphi \times \nabla h_k \cdot \mathbf{n}_k \ \mu \tag{83}$$

and hence the assumptions on η_k make Ω_k to be non-degenerate. The conditions of closure and absolute invariance of Ω_k implies

$$d_M \eta_k = 0 \ , \quad \eta_{k,t} + d_M i(v)(\eta_k) = 0 \tag{84}$$

the first of which makes Ω_k into a symplectic form. The second equation is obtained from

$$\mathcal{L}_{\partial_t + v}(\Omega_k) = di(\partial_t + v)(\rho_\varphi \eta_k \wedge dt) = \rho_\varphi(\eta_{k,t} + d_M i(v)(\eta_k)) \wedge dt \tag{85}$$

and expresses the advection of η_k by the flow of v. Eqs.(84) can also be realized as the integrability conditions for the equations

$$\eta_k = -d_M \varphi_k \ , \quad i(v)(\eta_k) = \varphi_{k,t} \tag{86}$$

defining the time-dependent function φ_k for given η_k. By the existence of these functions the suspended velocity field admits infinitely many symplectic formulations

$$i(\partial_t + v)(\Omega_k) = d\varphi_k \ , \quad k \geq 0 \tag{87}$$

which, for $k = 0, \omega_0 = \omega$ and $\varphi_k = \varphi$, coincide with the one we begin with. Since

$$\mathcal{L}_{\partial_t+v}(\varphi_k dt) = d\varphi_k + i(\partial_t + v)(d_M\varphi_k \wedge dt) = (\varphi_{k,t} + v(\varphi_k))dt = 0 \tag{88}$$

by Eqs.(86), we have

$$\mathcal{L}_{\partial_t+v}(\tilde{\theta}_k) = \mathcal{L}_{\partial_t+v}(\theta_k) \tag{89}$$

and hence the invariance class of $\tilde{\theta}_k$ is determined by that of θ_k in (59). For the extended form of the helicity conservation laws we compute

$$\begin{aligned} 0 &= -d(\theta_k \wedge \Theta_l) \ = \ -d[(\varphi_k dt - \tilde{\theta}_k) \wedge (\Omega_l - \eta_l \wedge dt)] & (90) \\ &= d(\tilde{\theta}_k \wedge \Omega_l) + (\Omega_k - \Theta_k) \wedge \Omega_l + \Omega_k \wedge \eta_l \wedge dt & (91) \\ &= d(\tilde{\theta}_k \wedge \Omega_l) + \Omega_k \wedge \Omega_l \quad \bullet & (92) \end{aligned}$$

Thus, the extentions Ω_k of degenerate, exact two-forms Θ_k is induced by a translation of the scalar part of the potential one-forms θ_k with a conserved function of the velocity field

$$\psi_k \mapsto \psi_k + \varphi_k \ \Rightarrow \ \theta_k \mapsto \tilde{\theta}_k \ , \quad \Theta_k = d\theta_k \mapsto \Omega_k = -d\tilde{\theta}_k \ . \tag{93}$$

The canonical one-forms $\tilde{\theta}_k$ are relative invariants of v

$$\mathcal{L}_{\partial_t+v}(\tilde{\theta}_k) = d\varphi_k \tag{94}$$

provided the gauge fixing conditions $\psi_k + \mathbf{v} \cdot \mathbf{A}_k = 0$ hold and they become absolute invariants whenever φ_k's are constants, that is, on the level surfaces of the Hamiltonian functions in Eqs.(87). The three-forms $\tilde{\theta}_k \wedge \Omega_l$ are also relative invariants

$$\mathcal{L}_{\partial_t+v}(\tilde{\theta}_k \wedge \Omega_l) = d(\varphi_k \Omega_l) \tag{95}$$

which when $\varphi_k = constant$ become absolute invariants because Ω_l are symplectic. In this case, the helicity densities are Lagrangian invariants. This is the particular relation (6) between invariants, geometric structures and

conservation laws. Below we shall consider a framework for geometric structures on $I \times M$ more general than the symplectic one to obtain the relation (5) between relative invariances and Eulerian conservation laws. The corresponding Lie algebraic structure on $C^{\infty}(I \times M)$ is a generalization of the Poisson bracket algebra to the one which relaxes the Leibniz' rule and this is connected with Jacobi structures on $I \times M$.

4.2 Jacobi structures

A local Lie algebra structure on the space $C^{\infty}(N)$ of smooth functions on a smooth manifold N is defined by a bilinear mapping

$$\{\ ,\ \} : C^{\infty}(N) \times C^{\infty}(N) \rightarrow C^{\infty}(N) \tag{96}$$

satisfying the conditions of skew-symmetry $\{f, g\} = -\{g, f\}$ and the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{97}$$

for arbitrary $f, g, h \in C^{\infty}(N)$. The bracket is local in the sense that

$$support(\{f, g\}) \subseteq support(f) \cap support(g) \tag{98}$$

and in general, $\{\ ,\ \}$ is not a derivation in its arguments. The local Lie algebra structure on $C^{\infty}(N)$ is linked with the Jacobi structure (Λ, E) on N through

$$\{f, g\} = \Lambda(df \wedge dg) + E(fdg - gdf) \tag{99}$$

where the bi-vector field $\Lambda : N \rightarrow \Lambda^2(TN) = TN \wedge TN$ and the vector field E on N satisfy the conditions

$$[\Lambda, \Lambda] = 2E \wedge \Lambda\ , \quad [\Lambda, E] = 0 \tag{100}$$

imposed by the Jacobi identity (97) [43],[44],[38]. The coordinate expression of the bracket

$$[\Lambda, \Lambda] = (\Lambda^{ij}_{,l}\Lambda^{lk} + \Lambda^{jk}_{,l}\Lambda^{li} + \Lambda^{ki}_{,l}\Lambda^{lj})\ \partial_i \wedge \partial_j \wedge \partial_k \tag{101}$$

for a bi-vector $\Lambda = \Lambda^{ij}\partial_i \wedge \partial_j/2$ is familiar from the Jacobi identity for a Poisson tensor and, $[\Lambda, E] = \mathcal{L}_E(\Lambda)$ is the Lie derivative [7],[29].

The Jacobi structure (Λ, E) on N includes, as a particular case, the Poisson structure Λ if $E = 0$. When Λ is of maximal rank on an even dimensional manifold N, one can define the two-form $\Omega \equiv \Lambda^{-1}$ and the one-form $\alpha \equiv i(E)(\Omega)$ satisfying the equations

$$d\Omega = \alpha \wedge \Omega \; , \qquad d\alpha = 0 \tag{102}$$

corresponding to Eqs.(100). The pair (Ω, α) is called a conformally symplectic structure on N and it reduces to a symplectic structure Ω whenever $\alpha = 0$. Since Ω is non-degenerate this is the same as $E = 0$ [43],[38].

Proposition 10 *Let θ_k be a relative invariant so that $\varphi_k \neq constant$. Then, for each l the pair*

$$\Omega_{kl} \equiv \varphi_k \Omega_l \; , \quad \alpha_k \equiv d log \varphi_k \tag{103}$$

defines a conformally symplectic structure on $I \times M$ and an isomorphisms from $C^\infty(I \times M)$ into vector fields on $I \times M$. For a function f, the vector field V_{klf} assigned by (103) corresponds to the Hamiltonian vector field for the function f/φ_k of the symplectic structure Ω_l.

Proof: Since Ω_k's are closed Ω_{kl}'s satisfy the conditions (102) with α_k given as in (103). To obtain the algebraic consequences we shall work with contravariant objects. The bi-vector dual to Ω_k can be computed to be

$$P_k = -(\mathbf{W}_k \cdot \mathbf{n}_k)^{-1} [\mathbf{W}_k \cdot \nabla \wedge \partial_t + +(\mathbf{W}_k \times \mathbf{v} - \rho_\varphi^{-1} \mathbf{n}_k) \cdot \nabla \wedge \nabla] \tag{104}$$

and one finds that the contravariant version of (102) is the Jacobi structure defined by the pair

$$P_{kl} \equiv \frac{1}{\varphi_k} P_l \; , \qquad E_{kl} \equiv -\frac{1}{\varphi_k^2} P_l(d\varphi_k) \tag{105}$$

which is conformally equivalent to P_l. The pair (105) satisfy Eqs.(100) and hence the brackets

$$\begin{aligned} \{f, g\}_{kl} &= P_{kl}(df \wedge dg) + E_{kl}(f dg - g df) && (106) \\ &= \frac{1}{\varphi_k} \{f, g\}_l - \frac{f}{\varphi_k^2} \{\varphi_k, g\}_l + \frac{g}{\varphi_k^2} \{\varphi_k, g\}_l && (107) \end{aligned}$$

where $\{\;,\;\}_l$ is the Poisson bracket defined by the bi-vector (104), fulfill the Jacobi identity (97) for each pair (k, l).

The Jacobi structure also provides an isomorphism between the Lie algebra of vector fields on $I \times M$ and the algebra of functions on $I \times M$ with the local bracket (106). If we let V_{klf} denote the vector field corresponding to the function f defined by the Jacobi structure (105) indexed by (k,l), then we find

$$V_{klf} = P_{kl}(df) + fE_{kl} = \frac{1}{\varphi_k}P_l(df) - \frac{f}{\varphi_k^2}P_l(d\varphi_k) = P_l(d(f/\varphi_k)) = V_{l(f/\varphi_k)} \tag{108}$$

which is the Hamiltonian vector field for the function f/φ_k defined by the Poisson bi-vector P_l. Note that the vector field E_{kl} is the Hamiltonian vector field for the function $f = 1$. •

Thus, in addition to the symplectic structures Ω_k, we have the families Ω_{kl} of conformally symplectic structures on $I \times M$ that coincide with Ω_l on the level surfaces $\varphi_k = \textit{constant}$ of Hamiltonian function on which Eulerian conserved densities become Lagrangian invariants. We therefore conclude that the absolute invariance of the canonical one-form, the degeneration of Eulerian conservation laws into Lagrangian invariants and, the conformal equivalence of the local structures (103) to the symplectic structures Ω_l as well as of their contravariant versions in Eqs.(105) and (104) are all the same. This verifies the statement of proposition (8).

5 Summary, discussions and conclusions

We shall summarize the main constructions of the last two sections, discuss the results and compare them with other works on helicity conservations. We shall indicate some generalizations as well as prospectives for a dynamical interpretation of helicity invariants.

5.1 Summary

Given the frozen in fields B and φ, the Eulerian dynamical equations imply that $\partial_t + v$ is Hamiltonian with

$$\Omega = \omega + (i(v)(\omega) - d_M\varphi) \wedge dt \; .$$

The left invariant generators W_k of particle relabelling symmetries can be obtained from infinitesimal Hamiltonian automorphisms of Ω. The Hamilto-

nian structure on M of W_k's is defined by the Nambu-Poisson bracket

$$\{f,g\}_\varphi = (\rho_\varphi)^{-1}\nabla\varphi \cdot \nabla f \times \nabla g \ .$$

For each k, $i(W_k)(\mu_M) = -\omega_k$ is a closed two-form on M. These can be extended to exact, degenerate two-forms

$$\Theta_k = \omega_k + i(v)(\omega_k) \wedge dt = d\theta_k \ , \quad -\theta_k = \psi_k dt + A_k \tag{109}$$

on $I \times M$ which are absolutely invariant under the flow of v. The potential one-forms θ_k can be classified in accordance with their orientation and invariance with respect to the flow of v. Those which are absolutely invariant give the Lagrangian

$$\frac{\partial \mathcal{H}_{kl}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{H}_{kl} = 0 \tag{110}$$

while the relatively invariant ones imply the Eulerian

$$\frac{\partial \mathcal{H}_{kl}}{\partial t} + \nabla \cdot [\mathcal{H}_{kl}\mathbf{v} - \rho_\varphi \varphi_k \mathbf{W}_l)] = 0 \tag{111}$$

conservation laws for the generalized helicity densities

$$\mathcal{H}_{kl} = \rho_\varphi \mathbf{a}_k \cdot \mathbf{W}_l = \rho_\varphi (W_l(\lambda_k) - \mathbf{A}_k \cdot \mathbf{W}_l) \ .$$

The densities satisfying Eqs.(110) and (111) can be parametrized, apart from the discrete parameters k, l, by functions in $ker(\partial_t + v)$ and $C^\infty(I \times M)/ker(\partial_t + v)$, respectively. The invariant forms defining the conservation laws admit extentions

$$\psi_k \mapsto \psi_k + \varphi_k \ \Rightarrow \ \theta_k \mapsto \tilde{\theta}_k \ , \quad \Theta_k = d\theta_k \mapsto \Omega_k = -d\tilde{\theta}_k \ . \tag{112}$$

to symplectic forms on $I \times M$. The conservation laws are expressed by the identity $d(\tilde{\theta}_k \wedge \Omega_l) + \Omega_k \wedge \Omega_l \equiv 0$. When $\tilde{\theta}_k$ is relatively invariant, this gives an Eulerian conservation law and the pair $\varphi_k \Omega_l$, $\alpha_k \equiv dlog\varphi_k$ defines a conformally symplectic structure. On the hypersurfaces $\varphi_k = constant$ these degenerate into a Lagrangian conservation law and the symplectic structure Ω_l, respectively.

5.2 Discussions and prospectives

We constructed helicity conservation laws from the invariant differential forms associated with the particle relabelling symmetries. This can be continued by introducing new families of invariant forms. For example, since W_k's commute with $\partial_t + v$ we can construct invariant two-forms by taking Lie derivatives of Ω with respect to W_k's. This we can compute using $\mathcal{L}_{U_k}(\Omega) = 0$ which follows from the fact that U_k's are Hamiltonian. Using the decomposition (18) and the Hamilton's equations for $\partial_t + v$ we get

$$\mathcal{L}_{W_k}(\Omega) = d\varphi \wedge d\xi_k \equiv \Sigma_k \tag{113}$$

which means that Ω is a relative invariant for W_k, $\forall k$. The degenerate two-forms Σ_k are absolute invariants for $\partial_t + v$ and are exact $\Sigma_k = d\sigma_k$. The gauge group $C^\infty(I \times M)$ also enters into the definition of Σ_k's and one can proceed as above to construct helicity type conservation laws expressed as closure of three-forms $\sigma_k \wedge \Sigma_l, \theta_k \wedge \Sigma_l$ and $\sigma_k \wedge \Theta_l$.

The helicity invariant which is first discovered in [45] have been studied in Refs. [46]-[49],[26],[36],[23] in the context of Noether theorems. The ergodic and topological interpretations of helicity type invariants for three-dimensional flows were introduced and investigated in Refs. [50]-[52],[10],[8]. A relation between infinite families of (magnetic) helicity invariants and magnetic surfaces has been remarked in Ref. [53]. The present construction inherits geometric objects for investigation of coadjoint orbit invariants. The framework can also be exploited for studying the interplay between these invariants each of which has been discussed separately in various contexts. For example, it offers W_k's for linking numbers, ω_k's for Hopf invariants, Θ_k's for Novikov type invariants and the Godbillon-Vey type invariants arises from the nilpotent generators of volume preserving diffeomorphisms.

We presented the kinematical aspects of helicity invariants in the geometric language of Jacobi structures. The results of proposition (10) are suggestive for further investigation, in the context of divergence-free vector fields, of the relations between Hamiltonian vector fields isomorphic to a local Lie algebra and Hamiltonian vector fields isomorphic to a Poisson bracket algebra. There is also a dynamical content of helicity invariants which will be presented in a forthcoming article. This is connected with Liouville structures on $I \times M$. A Liouville structure can be defined by a one-form together with an action of the multiplicative group R_* of non-zero real numbers [38].

In the present context the one-form is precisely the canonical one-form θ of the symplectic structure. The action of R_* is generated by the vector field which is dual to the three-form $\theta \wedge \Omega$ with respect to the symplectic volume. The dynamical properties of the fluid, such as viscosity, are implicit in this generator. Its divergence gives the evolution equation for the helicity density. This reduces to a conservation law for inviscid flows. Its action by Lie derivative corresponds to scaling transformations.

5.3 Conclusions

The symplectic structure on $I \times M$ provided us not only the way to construct infinitely many helicity type conservation laws associated with the Lie algebra of divergence free vector fields but also a kinematical interpretation of them with the Lie algebraic structures on function spaces over finite dimensional manifold $I \times M$. In Ref. [54] a similar interpretation with the local Lie algebras of Hamiltonian systems of hydrodynamic type, as introduced in Refs. [55] and [56], was described in the general framework of infinite dimensional Lie algebras.

The type of conservation laws associated with the particle relabelling symmetries and, in particular, the construction of infinitely many helicity type invariants for three dimensional flows seem to be much related to and rely on the conformal properties of the space of trajectories. The present framework incorporates the conformal transformations, which are not contained in $Diff_{vol}(M)$, into the study of kinematical invariants in connection with the algebraic structures on function spaces. We showed that Jacobi structures being associated with local Lie algebras provides a framework for investigation of properties which can not be obtained from the geometry of $Diff_{vol}(M)$.

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